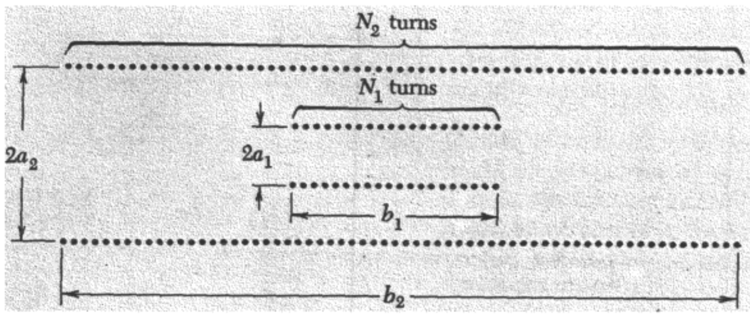
University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 11
Solutions by P. Pebler

1 *Purcell 7.21 A solenoid of radius a_1 and length b_1 is located inside a longer solenoid of radius a_2 and length b_2. The total number of turns is N_1 in the inner coil and N_2 on the outer. Work out a formula for the mutual inductance M.*



The mutual inductances M_{12} and M_{21} are equal, so we are free to calculate the inductance with whatever coil is more convenient. We find the flux through the inner coil. If we assume $b_2 \gg b_1$, the field through coil 1 will be fairly uniform and with the sign conventions shown,

$$\mathbf{B}_2 = \frac{\mu_o N_2 I_2}{b_2}\hat{\mathbf{x}} \quad ,$$

along the common axis of both coils. Since there are N_1 loops in this coil, the flux through all of them is

$$\phi_{12} = \pi a_1^2 N_1 \frac{\mu_o N_2 I_2}{b_2} \quad ,$$

and the induced emf is

$$\mathcal{E}_{12} = -\frac{d}{dt}\Phi_{12} = -\frac{\mu_o \pi a_1^2 N_1 N_2}{b_2}\frac{dI_2}{dt} \quad ,$$

and the mutual inductance is

$$M = \frac{\mu_o \pi a_1^2 N_1 N_2}{b_2} \quad .$$

2 *Purcell 7.22 A thin ring of radius a carries a static charge q. This ring is in a magnetic field of strength B_o, parallel to the ring's axis, and is supported so that it is free to rotate about that axis. If the field is switched off, how much angular momentum will be added to the ring? If the ring has mass m, show that it will acquire an angular velocity $\omega = qB_o/2mc$.*

We'll assume the charge is uniformly distributed around the ring, with linear density $\lambda = q/2\pi a$. Then the torque about the z axis is

$$\tau_z = \int d\tau_z = \int a\lambda \mathbf{f}\cdot d\mathbf{l} = \frac{q}{2\pi}\int \mathbf{E}\cdot d\mathbf{l} \ = \frac{q}{2\pi}\mathcal{E} \ .$$

The emf is

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi \quad ,$$

so that

$$\tau_z = \frac{dL_z}{dt} = -\frac{q}{2\pi c}\frac{d\Phi}{dt} \quad ,$$

and

$$L_{zf} - L_{zi} = -\frac{q}{2\pi c(\Phi_f - \Phi_i)} = \frac{q}{2\pi c}\pi a^2 B_o \quad ,$$

$$L_{zf} = \frac{qa^2 B_o}{2c} \quad .$$

The moment of inertia of the ring is $I = ma^2$, and

$$L_{zf} = I\omega = ma^2\omega = \frac{qa^2 B_o}{2c} \quad ,$$

$$\omega = \frac{qB_o}{2mc} \quad ,$$

equal to half the cyclotron frequency of a particle with mass m and charge q in a field B_0.

3 *Purcell 7.23* *There is evidence that a magnetic field exists in most of the interstellar space with a strength between* 10^{-6} *and* 10^{-5} *gauss. Adopting* 3×10^{-6} *gauss as a typical value, find the total energy stored in the magnetic field of the galaxy. Assume the galaxy is a disk roughly* 10^{23} *cm in diameter and* 10^{21} *cm thick. Assuming stars radiate about* 10^{44} *ergs/s, how many years of starlight is the magnetic energy worth?*

The magnetic energy is

$$U = \frac{1}{8\pi}\int B^2\, dV = \frac{1}{8\pi}(3 \times 10^{-6}\ gauss)^2(10^{21}\ cm)\pi(10^{23}/2\ cm)^2 = 3 \times 10^{54}\ ergs \quad ,$$

and this is

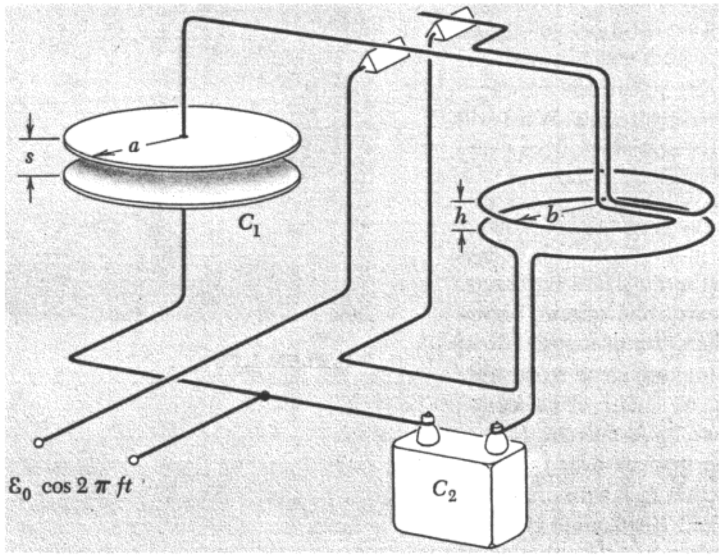
$$\frac{3 \times 10^{54}\ ergs}{10^{44}\ ergs/s} = 3 \times 10^{10}\ s = 900\ yr$$

of starlight.

4 *Purcell 7.29* *Consider the arrangement shown. The force between capacitor plates is balanced against the force between parallel wires. An alternating voltage of frequency* f *is applied to the capacitors* C_1 *and* C_2*. The charge flowing through* C_2 *constitutes the current through the rings. Suppose the time-average downward force on* C_2 *exactly balances the time averaged force on the wire loop. Show that under these conditions the constant c is*

$$c = (2\pi)^{3/2} a \left(\frac{b}{h}\right)^{1/2} \left(\frac{C_2}{C_1}\right) f \quad .$$

Assume $h \ll b$ *and ignore the self-inductance of the wire loop.*



The electric field in capacitor C_1 is

$$E = \frac{1}{s}\mathcal{E}_o \cos \omega t \quad ,$$

and the charge on it is

$$Q = C_1 V = C_1 \mathcal{E}_o \cos \omega t \quad .$$

The downward force will be the charge times the electric field due to the bottom plate. This field will be half of the total field.

$$F_1 = \frac{1}{2} E Q = \frac{\mathcal{E}_o^2 C_1}{2s} \cos^2 \omega t$$

Because the capacitance is

$$C_1 = \frac{\pi a^2}{4\pi s} = \frac{a^2}{4s} \quad ,$$

and the time average of $\cos^2$ is $1/2$, we may rewrite this as

$$\bar{F}_1 = \frac{\mathcal{E}_o^2 C_1^2}{a^2} \quad .$$

If we have $h \ll b$, we can use the force of two long wires. The field due to a wire is

$$B = \frac{2|I|}{cr} \quad ,$$

and the total force on the wire will be

$$F_2 = \frac{1}{c}|I|LB = \frac{1}{c} 2\pi b \frac{2I^2}{ch} = \frac{4\pi b I^2}{c^2 h} \quad .$$

The current is the time derivative of the charge on capacitor 2.

$$I = \frac{dQ_2}{dt} = \frac{d}{dt}(C_2 V) = -C_2 \mathcal{E}_o 2\pi f \sin \omega t$$

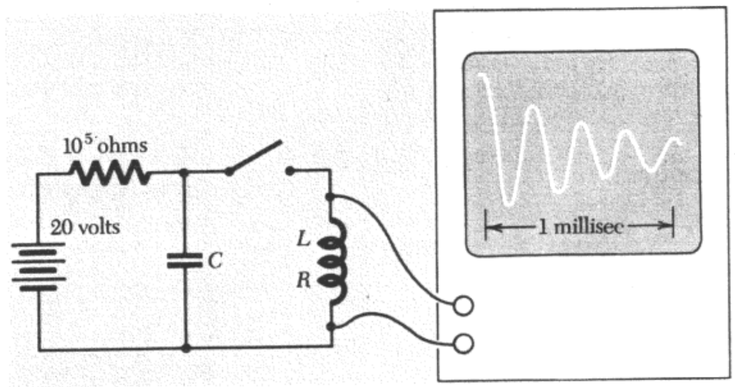
$$F_2 = \frac{4\pi b}{c^2 h} C_2^2 \mathcal{E}_o^2 (2\pi f)^2 \sin^2 \omega t$$

The time average of $\sin^2$ is also $1/2$ so

$$\bar{F}_2 = \frac{8\pi^3 b C_2^2 f^2 \mathcal{E}_o^2}{c^2 h} = \bar{F}_1 = \frac{C_1^2 \mathcal{E}_o^2}{a^2} \quad ,$$

$$c = (2\pi)^{3/2} a \left(\frac{b}{h}\right)^{1/2} \left(\frac{C_2}{C_1}\right) f \quad .$$

5 *Purcell 8.5* *The coil in the circuit shown in the diagram is known to have an inductance of* 0.01 *henry. when the switch is closed, the oscilloscope sweep is triggered. Determine the capacitance* C*. Estimate the value of the resistance* R *of the coil. What is the magnitude of the voltage across the oscilloscope input a long time, say 1 second after the switch has been closed?*



Parts (a.) and (b.) of this problem may be approximated by assuming that the battery is disconnected when the switch is closed. However, the problem actually is not too bad with the battery connected, so we will solve the original problem. The answers are the same except for the final voltage.

If you work out the equation for the charge on the capacitor C, you will find

$$L\frac{d^2Q}{dt^2} + \left(R_2 + \frac{L}{R_1 C}\right)\frac{dQ}{dt} + \left(\frac{R_1 + R_2}{R_1}\right)\frac{Q}{C} = \frac{VR_2}{R_1} \quad .$$

If we assume that the resistance R_2 of the inductor is much less that R_1, this becomes the LCR circuit equation. From the trace we see that

$$\omega = \frac{2\pi \cdot 4}{10^{-3}\ s} = 8\pi \times 10^3\ Hz \quad .$$

For low damping the capacitance is approximately

$$C = \frac{1}{L\omega^2} = 1.6 \times 10^{-7}\ F \quad .$$

Also from the trace, the amplitude falls off by a factor of e in about $0.5 \times 10^{-3}\ s$.

$$e^{-Rt/2L} = e^{-1}$$

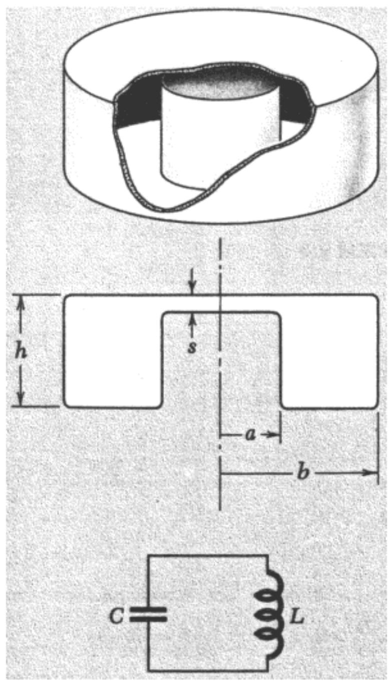
$$R = \frac{2L}{t} = 40\ ohms$$

If we wait one second, a long time, things will settle so that a steady current passes through the inductor and the voltage across it will be due to the resistance. If the current is I, the voltage is $V_2 = IR_2$ and

$$20\ V = I(10^5\ ohm + R_2) = V_2 \left(\frac{10^5 + 40}{40} \right) \quad ,$$

$$V = 8\ mV \quad .$$

6 *Purcell 8.7 A resonant cavity of the form illustrated is an essential part of many microwave oscillators. It can be regarded as a simple LC circuit. The inductance is that of a toroid with one turn. Find an expression for the resonant frequency of this circuit and show by a sketch the configuration of the magnetic and electric fields.*



The inner narrow circle will act as the capacitor while the outer ring is the solenoid. For a single turn toroid, the inductance is

$$L = \frac{2h}{c^2} \ln \frac{b}{a} \quad ,$$

and for the parallel plate capacitor,

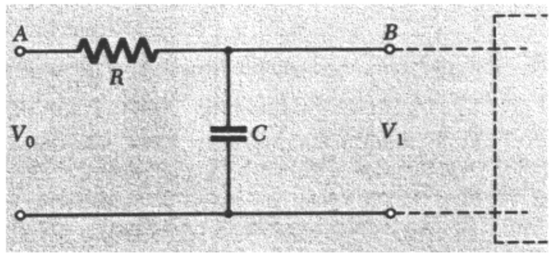
$$C = \frac{\pi a^2}{4\pi s} = \frac{a^2}{4s} \quad ,$$

so that

$$\omega = \sqrt{\frac{1}{LC}} = \frac{c}{a} \sqrt{\frac{2s}{h \ln(b/a)}} \quad .$$

The electric field is concentrated in the circular gap, where its direction is vertical; the magnetic field in the toroidal cavity is azimuthal in direction, with magnitude proportional to r^{-1}.

7 *Purcell 8.11* *An alternating voltage $V_o \cos \omega t$ is applied to the terminals at A. The terminals at B are connected to an audio amplifier of very high input impedance. Calculate the ratio $|V_1|^2/V_o^2$, where V_1 is the complex voltage at terminals B. Choose values for R and C to make $|V_1|^2/V_o^2 = 0.1$ for a 5000 hz signal. Show that for sufficiently high frequencies, the signal power is reduced by a factor 1/4 for every doubling of the frequency.*



Since the impedance on the right is very large, the impedance of the circuit is approximately

$$Z = R + \frac{1}{i\omega C} \quad ,$$

and the magnitude is

$$|Z| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \quad .$$

This gives us the magnitude of the complex current.

$$I_o = \frac{V_o}{|Z|} = \frac{V_o}{\sqrt{R^2 + 1/\omega^2 C^2}}$$

The impedance of just the capacitor is

$$Z_C = \frac{1}{i\omega C} \qquad |Z_C| = \frac{1}{\omega C} \quad .$$

This gives us the magnitude of the voltage across C.

$$|V_1| = I_o |Z_C| = \frac{V_o}{\sqrt{\omega^2 R^2 C^2 + 1}}$$

$$\frac{|V_1|^2}{V_o^2} = \frac{1}{\omega^2 R^2 C^2 + 1}$$

We would like an R and C such that

$$\frac{1}{[2\pi(5000\ hz)]^2 R^2 C^2 + 1} = 0.1 \quad ,$$

which can be done with many values of R and C, for example

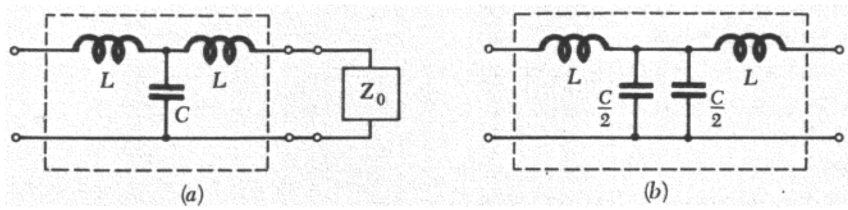
$$R = 100\ ohm \qquad C = 1\ \mu F \quad .$$

For sufficiently high frequencies we have

$$P \propto |V_1|^2 \propto \omega^{-2} \quad .$$

A reduction of signal power by a factor 8 rather than 4 per octave can be achieved by substituting an inductor L for the resistor R.

8 *Purcell 8.16* *An impedance Z_o is to be connected to the terminals on the right. For given frequency ω find the value which Z_o must have if the resulting impedance between the left terminals is Z_o. The required Z_o is a pure resistance R_o provided $\omega^2 < 2/LC$. What is Z_o in the special case $\omega = \sqrt{2/LC}$?*



We combine the impedances like resistances so that the total impedance is

$$Z = Z_L + \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L+Z_o}} \quad ,$$

with $Z_L = i\omega L$ and $Z_C = 1/i\omega C$. We set this equal to Z_o and simplify to obtain

$$Z_o = \sqrt{-\omega^2 L^2 + 2L/C} \quad .$$

This will be pure real and thus a pure resistance if

$$-\omega^2 L^2 + 2\frac{L}{C} > 0 \quad ,$$

$$\omega^2 < \frac{2}{LC} \quad .$$

In the special case $\omega = \sqrt{2/LC}$, we have $Z_o = 0$.